

# A first order wave equation for the electromagnetic radiation and the Dirac equation

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## Abstract

The classical electromagnetic theory dictates the well-known second order differential wave equation for the four-vector electromagnetic potential. Here we develop a heuristic first order wave equation for the electromagnetic radiation with shows formal similarity with the Dirac equation. The parallel leaves us to decompose the Dirac spinor on a set of two vector fields, by a particular identification of its components to the components of two four-potential-like magnitudes. As a result, we arrive to striking similarities with the formalism of the electromagnetic theory.

Keywords: electromagnetic wave; wave equation; Dirac equation; spinor field; vector field.

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## 1 Introduction

The classical electromagnetic theory provides the second order wave equation for the electromagnetic radiation (ER),  $\square\phi^\mu = 0$ , where  $\phi^\mu$  is the four-vector electromagnetic potential and  $\square$  is the d'Alembert operator.<sup>1,2</sup> As is well known,  $\phi^\mu$  is comprises the time-like scalar potential  $\phi_0$  and the space-like vector potential  $\vec{\phi}$ .

Under the viewpoint of the special relativity theory, the four-vector potential  $\phi^\mu$  plays the central role as the relativistically invariant magnitude, while the electric  $\mathbf{E}$  and the magnetic  $\mathbf{B}$  fields are only three-dimensional vectors that appear in the components of the relativistically invariant electromagnetic tensor.<sup>1,2</sup>

On the formalism of the second quantization of fields,  $\phi^\mu$  is a Fourier series decomposed in terms of the nonclassical creation and annihilation operators,<sup>3</sup>

and  $\phi^\mu$  is also considered to be the central root of the theory instead of  $\mathbf{E}$  and  $\mathbf{B}$ .

In the framework of quantum mechanics there is a clear distinction between the waves of matter and the physical waves of the ER. While the ER waves correspond to the physical fields  $\mathbf{E}$  and  $\mathbf{B}$ , the electron matter waves constitute only a probability field.<sup>6</sup> However, a consistent conception for a photon wave function was proposed recently<sup>7,8</sup> based on the three dimensional complex Riemann-Silberstein vector<sup>9</sup>  $\mathbf{G} = \mathbf{E} + i\mathbf{B}$  ( $c = 1$ ) and on the Maxwell equations, in which  $\mathbf{G}$  obeys the eigenvalue equation  $\hat{H}\mathbf{G} = i\hat{\mathbf{p}} \times \mathbf{G}$  in terms of the usual Hamiltonian  $\hat{H}$  and the momentum  $\hat{\mathbf{p}}$  operators, with energy eigenvalues  $E = \pm p$ . Under this perspective,  $\mathbf{G}$  is considered to be the photon wave function less of a factor, and the ER energy density  $|\mathbf{G}|^2 \propto u$  is considered to be equivalent to the photon probability density.<sup>10</sup> The four potential  $\phi^\mu$  loses its position as the central magnitude while the electric  $\mathbf{E}$  and the magnetic  $\mathbf{B}$  fields play the central role on through the complex vector  $\mathbf{G}$ .

The present brief work considers this relativism of the hierarchy of  $\phi^\mu$  and the electric and the magnetic fields  $\mathbf{E}$  and  $\mathbf{B}$  as the basis for the construction of a reasoning. We built an *ad hoc* heuristic first order differential equation for the waves of ER and show its similarity with the Dirac equation. We reach to the unexpected conclusion that the Dirac spinor  $\Psi$  is directly associated to two four vector fields, to which we associate “electric like” and “magnetic like fields” and find formal similarities with the electromagnetic theory.

## 2 The electromagnetic radiation and its first order wave equation

We start considering the fact that the electric and the magnetic fields of the ER are mutually perpendicular, *i.e.*

$$\mathbf{E} \cdot \mathbf{B} = 0 \quad \Leftrightarrow \quad E_x B_x + E_y B_y + E_z B_z = 0. \quad (1)$$

Note that  $\mathbf{E} \cdot \mathbf{B}$  is an invariant of the electromagnetism,<sup>2</sup> and relation 1 is valid in any reference frame.

For our purposes, let us consider a reference frame where the projections of  $\mathbf{E}$  and  $\mathbf{B}$  on the plane  $xy$  are mutually perpendicular:

$$E_x = B_y, \quad E_y = -B_x. \quad (2)$$

Note that this is a weak mathematical restriction that does not represent a problem under the physical aspect. In fact, if we choose any other reference frame  $S'$  moving with a velocity  $v$  with respect to the given reference frame  $S$ , we may guide the orientation of the new axis  $x', y', z'$  respectively parallel to  $x, y, z$ , to arrive to similar relations as 2 for the new components of the electric  $\mathbf{E}'$  and the magnetic  $\mathbf{B}'$  fields. The arbitrary choice of orientation of axis is neither incorrect nor a new issue, being commonly applied on the context of angular momentum and spin in quantum mechanics.

Expressions 1 and 2 imply that

$$E_z = 0 \quad \vee \quad B_z = 0 \quad . \quad (3)$$

Since  $\mathbf{E} = -\partial_0 \phi - \nabla \phi_0$  and  $\mathbf{B} = \nabla \times \phi$ , conditions 2 and the first one of 3 (for example) can be grouped as (adopting Einstein summation rule)

$$\left. \begin{aligned} -\partial_0 \phi_1 - \partial_1 \phi_0 - \epsilon_{2,jk} \partial_j \phi_k &= 0 \\ -\partial_0 \phi_2 - \partial_2 \phi_0 + \epsilon_{1,jk} \partial_j \phi_k &= 0 \\ -\partial_0 \phi_3 - \partial_3 \phi_0 &= 0 \end{aligned} \right\} \quad (4)$$

We consider also the Lorentz gauge  $\partial_\mu \phi^\mu = 0$ , and group this with equations 4 to form the matrix equation

$$\varepsilon^\mu \partial_\mu \phi = 0 \quad (5)$$

which is our *heuristic, first order linear homogeneous differential equation for the ER* (being  $\phi = (\phi^0, \phi^1, \phi^2, \phi^3)$  expressed as a column vector), where we have defined the four 4×4 matrices:

$$\begin{aligned} \varepsilon^0 &= \begin{pmatrix} 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad \varepsilon^1 = \begin{pmatrix} -1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \\ \varepsilon^2 &= \begin{pmatrix} 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \quad \varepsilon^3 = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \end{aligned} \quad (6)$$

whose asymmetry arises from the chosen orientation of the reference frame. The above *Ansatz* of adoption of the electric  $\mathbf{E}$  and the magnetic  $\mathbf{B}$  fields as the relevant magnitudes left to an equation that involves only the four potential  $\phi$ .

Under the mathematical aspect, Equation 5 determines the four-potential  $\phi^\mu$  from a set of boundary conditions (one set integration constants), while the traditional second order wave equation  $\square \phi^\mu = 0$  requires two sets of integration constants. In this sense, the traditional form  $\square \phi^\mu = 0$  can be considered in some sense “redundant”.

We stress that equation 5 is invariant under the Lorentz transform between the set of all the reference frames where 2 is valid. On the remaining reference frames where 2 and (consequently) 5 are not valid, we simply perform the necessary mathematical rotation of the axis, and these equations remain valid.

### 3 Dirac equation: from the spinor to vector fields

Here we notice the similarity between Equation 5 and the Dirac equation for massless fermions (here taking  $\hbar = 1$ ):<sup>2,6</sup>

$$i\gamma^\mu \partial_\mu \Psi = 0 \quad (7)$$

This similarity suggests applying to the Dirac equation 7 the inverse route of mathematical manipulation we followed to obtain 5. For that, we write explicitly the spinor  $\Psi = (\psi_n + i\chi_n)$ , where  $\psi_n, \chi_n \in \mathfrak{R}, n = 0, \dots, 3$  are the real

and the imaginary parts of each component of  $\Psi$ . By direct substitution on 7, we obtain the following set of equations:

$$\begin{pmatrix} \partial_2 \chi_3 + \partial_0 \psi_0 + \partial_3 \psi_2 + \partial_1 \psi_3 \\ -\partial_2 \chi_2 + \partial_0 \psi_1 + \partial_1 \psi_2 - \partial_3 \psi_3 \\ -\partial_2 \chi_1 - \partial_3 \psi_0 - \partial_1 \psi_1 - \partial_0 \psi_2 \\ \partial_2 \chi_0 - \partial_1 \psi_0 + \partial_3 \psi_1 - \partial_0 \psi_3 \end{pmatrix} = 0 \quad (8)$$

and

$$\begin{pmatrix} \partial_0 \chi_0 + \partial_3 \chi_2 + \partial_1 \chi_3 - \partial_2 \psi_3 \\ \partial_0 \chi_1 + \partial_1 \chi_2 - \partial_3 \chi_3 + \partial_2 \psi_2 \\ -\partial_3 \chi_0 - \partial_1 \chi_1 - \partial_0 \chi_2 + \partial_2 \psi_1 \\ -\partial_1 \chi_0 + \partial_3 \chi_1 - \partial_0 \chi_3 - \partial_2 \psi_0 \end{pmatrix} = 0 \quad (9)$$

Let us make the change of labeling:

$$\begin{pmatrix} \psi_0 \\ \psi_1 \\ \psi_2 \\ \psi_3 \end{pmatrix} \mapsto \begin{pmatrix} \psi_0 \\ -\chi_2 \\ \psi_3 \\ \psi_1 \end{pmatrix}; \quad \begin{pmatrix} \chi_0 \\ \chi_1 \\ \chi_2 \\ \chi_3 \end{pmatrix} \mapsto \begin{pmatrix} \chi_3 \\ \chi_1 \\ \chi_0 \\ \psi_2 \end{pmatrix} \quad (10)$$

Then, we define the three-dimensional vector magnitudes  $\mathbf{E}_{\psi\{\chi\}}$  and  $\mathbf{B}_{\psi\{\chi\}}$ ,

$$\begin{aligned} \mathbf{E}_\psi &= -\partial_0 \boldsymbol{\psi} - \nabla \psi_0; & \mathbf{B}_\psi &= \nabla \times \boldsymbol{\psi}; \\ \mathbf{E}_\chi &= -\partial_0 \boldsymbol{\chi} - \nabla \chi_0; & \mathbf{B}_\chi &= \nabla \times \boldsymbol{\chi}, \end{aligned} \quad (11)$$

which satisfy the same algebra of the usual electromagnetic fields  $\mathbf{E}$  and  $\mathbf{B}$  with respect to the four-vector electromagnetic potential  $\phi^\mu$ . The magnitudes  $\mathbf{E}_{\psi\{\chi\}}$  and  $\mathbf{B}_{\psi\{\chi\}}$  constitute real vectors on the three dimensional space as well as  $\boldsymbol{\psi}$  and  $\boldsymbol{\phi}$ . The substitution of these magnitudes in expressions 8 and 9 gives:

$$\begin{pmatrix} \partial_\mu \psi^\mu \\ E_{\chi,2} - B_{\psi,2} \\ E_{\psi,3} + B_{\chi,3} \\ E_{\psi,1} + B_{\chi,1} \end{pmatrix} = 0 \quad \text{and} \quad \begin{pmatrix} E_{\chi,3} - B_{\psi,3} \\ E_{\chi,1} - B_{\psi,1} \\ \partial_\mu \chi^\mu \\ E_{\psi,2} + B_{\chi,2} \end{pmatrix} = 0. \quad (12)$$

Equations 12 can be grouped in:

$$\mathbf{E}_\psi = -\mathbf{B}_\chi \quad \mathbf{E}_\chi = \mathbf{B}_\psi, \quad (13)$$

together with the two “gauge like” conditions:

$$\partial_\mu \chi^\mu = 0, \quad \partial_\mu \psi^\mu = 0 \quad (14)$$

We stress that relations 12 between the vector fields  $\mathbf{E}_{\psi\{\chi\}}, \mathbf{B}_{\psi\{\chi\}}$  are the equivalent, for the massless fermion, to the photon relations 2.

As we verified, the other alternative permuted relabeling instead of 10 leave also to the “gauge like” conditions 14 and to the fields  $\mathbf{E}_{\psi\{\chi\}}$  and  $\mathbf{B}_{\psi\{\chi\}}$ , apart from irrelevant changes of signal.

As a partial conclusion, we verified that the spinor field  $\Psi$  for massless fermions is equivalent to two four vector fields  $\chi^\mu$  and  $\psi^\mu$ , from which we extract the set of vectors  $\mathbf{E}_{\psi\{\chi\}}, \mathbf{B}_{\psi\{\chi\}}$  of the three dimensional space.

Now we treat the most general case of massive fermions, for which Dirac equation 7 has the mass term.<sup>2,6</sup> Explicitly, we have

$$\begin{pmatrix} \partial_\mu \psi^\mu \\ E_{\chi,2} - B_{\psi,2} \\ E_{\psi,3} + B_{\chi,3} \\ E_{\psi,1} + B_{\chi,1} \end{pmatrix} = m \begin{pmatrix} \chi_3 \\ \chi_1 \\ \chi_0 \\ \psi_2 \end{pmatrix} \quad (15)$$

and

$$\begin{pmatrix} E_{\chi,3} - B_{\psi,3} \\ E_{\chi,1} - B_{\psi,1} \\ \partial_\mu \chi^\mu \\ E_{\psi,2} + B_{\chi,2} \end{pmatrix} = m \begin{pmatrix} \psi_0 \\ -\chi_2 \\ \psi_3 \\ -\psi_1 \end{pmatrix}, \quad (16)$$

which were obtained respectively from the real and the imaginary parts of the Dirac massive fermion equation after the relabeling 10, respectively.

On the present case it is interesting to consider the components of the four vector current density  $j^\mu = \bar{\psi} \gamma^\mu \gamma^0 \psi$ , which can be written as

$$\left. \begin{aligned} j^0 &= m^{-2} (\mathbf{C}^2 + \mathbf{D}^2) + \psi_3^2 + \chi_3^2 \\ \mathbf{j} &= 2m^{-2} (\mathbf{C} \times \mathbf{D}) + \mathbf{\Lambda} \end{aligned} \right\} \quad (17)$$

The expressions in 17 were written in a compact form in terms of the three-dimensional vector  $\mathbf{\Lambda}$  with components

$$\left. \begin{aligned} \Lambda_{1,2} &= -(\boldsymbol{\psi} \times \boldsymbol{\chi})_{1,2} \\ \Lambda_3 &= (\chi_0 \chi_3 + \psi_0 \psi_3) \end{aligned} \right\} \quad (18)$$

and the three-dimensional vectors,

$$\mathbf{C} = \mathbf{E}_\chi - \mathbf{B}_\psi; \quad \mathbf{D} = \mathbf{E}_\psi + \mathbf{B}_\chi. \quad (19)$$

Note that for the particular case of massless fermions,  $\mathbf{C}$  and  $\mathbf{D}$  must vanish, which is a direct consequence of the equalities 13. On this case,  $j^0 \equiv \psi_3^2 + \chi_3^2$  and  $\mathbf{j} \equiv \mathbf{\Lambda}$ .

The expressions in 17 are strikingly similar to the expressions of the electromagnetic energy density  $u = (2\varepsilon_0)^{-1}(\mathbf{E}^2 + \mathbf{B}^2)$  and the Poynting vector  $\mathbf{S} = \varepsilon_0^{-1}(\mathbf{E} \times \mathbf{B})$  by the replacements  $\mathbf{C} \mapsto \mathbf{E}$ ,  $\mathbf{D} \mapsto \mathbf{B}$  and  $m^2 \mapsto 2\varepsilon_0$ . The present formalism differs by the emergence of the terms  $(\psi_3^2 + \chi_3^2)$  and  $\mathbf{\Lambda}$ , which have not correspondence on the electromagnetic theory.

## 4 Final comments

Despite the strong parallel with the electromagnetic theory, the real existence of the vectors  $\mathbf{E}_{\psi\{\chi\}}, \mathbf{B}_{\psi\{\chi\}}$  or either  $\mathbf{C}, \mathbf{D}$  as measurable magnitudes, should

be proven experimentally. However, presently there is not evidence of their association to a physical interaction that would enable either a direct or an indirect detection.

The emergence of four-vector magnitudes  $\pi^\mu = (\pi_0, \boldsymbol{\pi})$  associated with three-dimensional space vectors  $\mathbf{e} = -\nabla\pi_0 - \partial_0\boldsymbol{\pi}$  and  $\mathbf{b} = \nabla \times \boldsymbol{\pi}$  shows to be an intrinsic characteristic on different physical contexts. As well known, on the electromagnetic theory  $\pi^\mu$  is the four vector electromagnetic potential and  $\mathbf{e}$ ,  $\mathbf{b}$  are the electric and the magnetic field vectors. On the other side, the present work showed the same mathematical structure inside the Dirac equation. Either, this is verified also on the structure of the Riemann curvature tensor, in first order of approximation.<sup>11,12</sup>

Now, note that equation 5 emerged from the projection of the ER fields on the particular reference frame where the expressions 2 are valid. On the other side, the electric and the magnetic fields satisfy globally the Maxwell equations, which are the most general equations that govern these fields.

In this sense, we can speculate if equation 13 – and consequently the Dirac equation itself – correspond to a “fermionic radiation field” that are globally governed by most general field equations.

As a minor observation, the six Equations of 15 and 16 that does not contain the four divergencies  $\partial_\mu\psi^\mu$  and  $\partial_\mu\chi^\mu$  show a formal similarity with a classical continuum mechanics model of the dynamics of two fluids of mass density  $m$  governed by two acceleration fields  $\mathbf{a}_1 = (\psi_2, -\psi_1, \chi_0)$ ,  $\mathbf{a}_2 = (-\chi_2, \chi_1, \psi_0)$  under the forces  $\mathbf{F}_{1\{2\}} = \mathbf{E}_{\psi\{\chi\}} + \{\cdot\} \mathbf{B}_{\chi\{\psi\}}$ , such that  $\mathbf{F}_{1\{2\}} = m\mathbf{a}_{1\{2\}}$ . These equations can also be expressed on the complex vector equation

$$\mathbf{F} = m\mathbf{a} \quad (20)$$

where  $\mathbf{F} = \mathbf{G}_\psi - i\mathbf{G}_\chi$ , being  $\mathbf{G}_{\psi\{\chi\}} = \mathbf{E}_{\psi\{\chi\}} + i\mathbf{B}_{\psi\{\chi\}}$  and  $\mathbf{a} = (\chi_0 + i\chi_2, \psi_2 - i\psi_0, -\psi_1 - i\chi_1)$ .

In summary, we verified that the spinor field of massive or massless fermions may be written in terms of an equivalent to a set of two four vectors by an appropriate mathematical transform. The analysis revealed striking similarities with the framework of the electromagnetic theory.

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